

# $1^{++}$ Nonet Singlet-Octet Mixing Angle, Strange Quark Mass, and Strange Quark Condensate

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## Abstract

Two strategies are taken into account to determine the  $f_1(1420)$ - $f_1(1285)$  mixing angle  $\theta$ . (i) First, using the Gell-Mann-Okubo mass formula together with the  $K_1(1270)$ - $K_1(1400)$  mixing angle  $\theta_{K_1} = (-34 \pm 13)^\circ$  extracted from the data for  $\mathcal{B}(B \rightarrow K_1(1270)\gamma)$ ,  $\mathcal{B}(B \rightarrow K_1(1400)\gamma)$ ,  $\mathcal{B}(\tau \rightarrow K_1(1270)\nu_\tau)$ , and  $\mathcal{B}(\tau \rightarrow K_1(1420)\nu_\tau)$ , gave  $\theta = (23_{-23}^{+17})^\circ$ . (ii) Second, from the study of the ratio for  $f_1(1285) \rightarrow \phi\gamma$  and  $f_1(1285) \rightarrow \rho^0\gamma$  branching fractions, we have a two-fold solution  $\theta = (19.4_{-4.6}^{+4.5})^\circ$  or  $(51.1_{-4.6}^{+4.5})^\circ$ . Combining these two analyses, we thus obtain  $\theta = (19.4_{-4.6}^{+4.5})^\circ$ . We further compute the strange quark mass and strange quark condensate from the analysis of the  $f_1(1420)$ - $f_1(1285)$  mass difference QCD sum rule, where the operator-product-expansion series is up to dimension six and to  $\mathcal{O}(\alpha_s^3, m_s^2\alpha_s^2)$  accuracy. Using the average of the recent lattice results and the  $\theta$  value that we have obtained as inputs, we get  $\langle\bar{s}s\rangle/\langle\bar{u}u\rangle = 0.41 \pm 0.09$ .

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## I. INTRODUCTION

The  $f_1(1285)$  and  $f_1(1420)$  mesons with quantum number  $J^{PC} = 1^{++}$  are the members of the  $1^3P_1$  states in the quark model language, and are mixtures of the pure octet  $f_8$  and singlet  $f_1$ , where the mixing is characterized by the mixing angle  $\theta$ . The BaBar results for the upper bounds of  $B^- \rightarrow f_1(1285)K^-, f_1(1420)K^-$  were available recently [1]. The relative ratio of these two modes is highly sensitive to  $\theta$  [2]. On the other hand, in the two-body  $B$  decay involving the  $K$  meson in the final state, the amplitude receives large corrections from the chiral enhancement  $a_6$  term which is inversely proportional to the strange-quark mass. The quark mass term mixes left- and right-handed quarks in the QCD Lagrangian. The spontaneous breaking of chiral symmetry from  $SU(3)_L \times SU(3)_R$  to  $SU(3)_V$  is further broken by the quark masses  $m_{u,d,s}$  when the baryon number is added to the three commuting conserved quantities  $Q_u, Q_d$ , and  $Q_s$ , respectively, the numbers of  $q - \bar{q}$  quarks for  $q = u, d$ , and  $s$ . The nonzero quark condensate which signals dynamical symmetry breaking is the important parameter in QCD sum rules [3], while the magnitude of the strange quark mass can result in the flavor symmetry breaking in the quark condensate. In an earlier study  $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle \sim 0.8 < 1$  was usually taken. However, very recently the Jamin-Lange approach [4] together with the lattice result for  $f_{B_s}/f_B$  [5] and also the Schwinger-Dyson equation approach [6] can give a central value larger than 1.

In this paper, we shall embark on the study of the  $f_1(1420)$  and  $f_1(1285)$  mesons to determine the mixing angle  $\theta$ , strange quark mass, and strange quark condensate. In Sec. II, we shall present detailed discussions on the determination of the mixing angle  $\theta$ . Substituting the  $K_1(1270)$ - $K_1(1400)$  mixing angle, which was extracted from the  $B \rightarrow K_1\gamma$  and  $\tau \rightarrow K_1\nu_\tau$  data, to the Gell-Mann-Okubo mass formula, we can derive the value of  $\theta$ . Alternatively, from the analysis of the decay ratio for  $f_1(1285) \rightarrow \phi\gamma$  and  $f_1(1285) \rightarrow \rho^0\gamma$ , we have a more accurate estimation for  $\theta$ . In Sec. III we shall obtain the mass difference QCD sum rules for the  $f_1(1420)$  and  $f_1(1285)$  to determine the magnitude of the strange quark mass. From the sum rule analysis, we obtain the constraint ranges for  $m_s$  and  $\theta$  as well as for  $\langle \bar{s}s \rangle$ . Many attempts have been made to compute  $m_s$  using QCD sum rules and finite energy sum rules [7–13]. The running strange quark mass in the  $\overline{\text{MS}}$  scheme at a scale of  $\mu \approx 2$  GeV is  $m_s = 101^{+29}_{-21}$  MeV given in the particle data group (PDG) average [14]. More precise lattice estimates have been recently obtained as  $m_s(2 \text{ GeV}) = 92.2(1.3)$  MeV in [15],  $m_s(2 \text{ GeV}) = 96.2(2.7)$  MeV in [16], and  $m_s(2 \text{ GeV}) = 95.1(1.1)(1.5)$  MeV in [17]. These lattice results agree with strange scalar/pseudoscalar sum rule results which are  $m_s \simeq 95(15)$  MeV. In the present study, we study the  $m_s$  from a new frame, the  $f_1(1420)$ - $f_1(1285)$  mass difference sum rule, which may result in larger uncertainties due to the input parameters. Nevertheless, it can be a crosscheck compared with the previous studies. Further using the very recent lattice result for  $m_s(2 \text{ GeV}) = 93.6 \pm 1.0$  MeV as the input, we obtain an estimate for the strange quark condensate.

## II. SINGLET-OCTET MIXING ANGLE $\theta$ OF THE $1^{++}$ NONET

### A. Definition

In the quark model,  $a_1(1260)$ ,  $f_1(1285)$ ,  $f_1(1420)$ , and  $K_{1A}$  are classified in  $1^{++}$  multiplets, which, in terms of spectroscopic notation  $n^{2S+1}L_J$ , are  $1^3P_1$   $p$ -wave mesons. Analogous to  $\eta$  and  $\eta'$ , because of SU(3) breaking effects,  $f_1(1285)$  and  $f_1(1420)$  are the mixing states of the pure octet  $f_8$  and singlet  $f_1$ ,

$$|f_1(1285)\rangle = |f_1\rangle \cos \theta + |f_8\rangle \sin \theta, \quad |f_1(1420)\rangle = -|f_1\rangle \sin \theta + |f_8\rangle \cos \theta. \quad (1)$$

In the present paper, we adopt

$$f_1 = \frac{1}{\sqrt{3}}(\bar{u}u + \bar{d}d + \bar{s}s), \quad (2)$$

$$f_8 = \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s), \quad (3)$$

where there is a relative sign difference between the  $\bar{s}s$  contents of  $f_1$  and  $f_8$  in our convention. From the Gell-Mann-Okubo mass formula, the mixing angle  $\theta$  satisfies

$$\cos^2 \theta = \frac{3m_{f_1(1285)}^2 - (4m_{K_{1A}}^2 - m_{a_1}^2)}{3(m_{f_1(1285)}^2 - m_{f_1(1420)}^2)}, \quad (4)$$

where

$$m_{K_{1A}}^2 = \langle K_{1A} | \mathcal{H} | K_{1A} \rangle = m_{K_1(1400)}^2 \cos^2 \theta_{K_1} + m_{K_1(1270)}^2 \sin^2 \theta_{K_1}, \quad (5)$$

with  $\mathcal{H}$  being the Hamiltonian. Here  $\theta_{K_1}$  is the  $K_1(1400)$ - $K_1(1270)$  mixing angle. The sign of the mixing angle  $\theta$  can be determined from the mass relation [14]

$$\tan \theta = \frac{4m_{K_{1A}}^2 - m_{a_1}^2 - 3m_{f_1(1420)}^2}{3m_{18}^2}, \quad (6)$$

where  $m_{18}^2 = \langle f_1 | \mathcal{H} | f_8 \rangle \simeq (m_{a_1}^2 - m_{K_{1A}}^2)2\sqrt{2}/3 < 0$ , we find  $\theta > 0$ . Because of the strange and nonstrange light quark mass differences,  $K_{1A}$  is not the mass eigenstate and it can mix with  $K_{1B}$ , which is one of the members in the  $1^1P_1$  multiplets. From the convention in [18] (see also discussions in [19, 20]), we write the two physical states  $K_1(1270)$  and  $K_1(1400)$  in the following relations:

$$\begin{aligned} |K_1(1270)\rangle &= |K_{1A}\rangle \sin \theta_K + |K_{1B}\rangle \cos \theta_K, \\ |K_1(1400)\rangle &= |K_{1A}\rangle \cos \theta_K - |K_{1B}\rangle \sin \theta_K. \end{aligned} \quad (7)$$

The mixing angle was found to be  $|\theta_{K_1}| \approx 33^\circ, 57^\circ$  in [18] and  $\approx \pm 37^\circ, \pm 58^\circ$  in [21]. A similar range  $35^\circ \lesssim |\theta_{K_1}| \lesssim 55^\circ$  was obtained in [22]. The sign ambiguity for  $\theta_{K_1}$  is due to the fact that one can add arbitrary phases to  $|\bar{K}_{1A}\rangle$  and  $|\bar{K}_{1B}\rangle$ . This sign ambiguity can be removed by fixing the signs of decay constants  $f_{K_{1A}}$  and  $f_{\bar{K}_{1B}}^\perp$ , which are defined by

$$\langle 0 | \bar{\psi} \gamma_\mu \gamma_5 s | \bar{K}_{1A}(P, \lambda) \rangle = -i f_{K_{1A}} m_{K_{1A}} \epsilon_\mu^{(\lambda)}, \quad (8)$$

$$\langle 0 | \bar{\psi} \sigma_{\mu\nu} s | \bar{K}_{1B}(P, \lambda) \rangle = i f_{K_{1B}}^\perp \epsilon_{\mu\nu\alpha\beta} \epsilon_{(\lambda)}^\alpha P^\beta, \quad (9)$$

where  $\epsilon^{0123} = -1$  and  $\psi \equiv u$  or  $d$ . Following the convention in [20], we adopt  $f_{K_{1A}} > 0$ ,  $f_{K_{1B}}^\perp > 0$ , so that  $\theta_{K_1}$  should be negative to account for the observable  $\mathcal{B}(B \rightarrow K_1(1270)\gamma) \gg \mathcal{B}(B \rightarrow K_1(1400)\gamma)$  [23, 24]. Furthermore, from the data of  $\tau \rightarrow K_1(1270)\nu_\tau$  and  $K_1(1400)\nu_\tau$  decays together with the sum rule results for the  $K_{1A}$  and  $K_{1B}$  decay constants, the mixing angle  $\theta_{K_1} = (-34 \pm 13)^\circ$  was obtained in [24]. Substituting this value into (4), we then obtain  $\theta^{\text{quad}} = (23_{-23}^{+17})^\circ$  [25], i.e.,  $\theta^{\text{quad}} = 0^\circ - 40^\circ$ <sup>1</sup>.

## B. The determination of $\theta$

Experimentally, since  $K^*\bar{K}$  and  $K\bar{K}\pi$  are the dominant modes of  $f_1(1420)$ , whereas  $f_0(1285)$  decays mainly to the  $4\pi$  states, this suggests that the quark content is primarily  $s\bar{s}$  for  $f_1(1420)$  and  $n\bar{n} = (u\bar{u} + d\bar{d})/\sqrt{2}$  for  $f_1(1285)$ . Therefore, the mixing relations can be rewritten to exhibit the  $n\bar{n}$  and  $s\bar{s}$  components which decouple for the ideal mixing angle  $\theta_i = \tan^{-1}(1/\sqrt{2}) \simeq 35.3^\circ$ . Let  $\bar{\alpha} = \theta_i - \theta$ , we rewrite these two states in the flavor basis<sup>2</sup>,

$$\begin{aligned} f_1(1285) &= \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d) \cos \bar{\alpha} + \bar{s}s \sin \bar{\alpha}, \\ f_1(1420) &= \frac{1}{\sqrt{2}}(\bar{u}u + \bar{d}d) \sin \bar{\alpha} - \bar{s}s \cos \bar{\alpha}. \end{aligned} \quad (10)$$

Since the  $f_1(1285)$  can decay into  $\phi\gamma$ , we know that  $f_1(1285)$  has the  $s\bar{s}$  content and  $\theta$  deviates from its ideal mixing value. To have a more precise estimate for  $\theta$ , we study the ratio of  $f_1(1285) \rightarrow \phi\gamma$  and  $f_1(1285) \rightarrow \rho^0\gamma$  branching fractions. Because the electromagnetic (EM) interaction Lagrangian is given by

$$\begin{aligned} \mathcal{L}_I &= -A_{\text{EM}}^\mu (e_u \bar{u} \gamma_\mu u + e_d \bar{d} \gamma_\mu d + e_s \bar{s} \gamma_\mu s) \\ &= -A_{\text{EM}}^\mu \left( (e_u + e_d) \frac{\bar{u} \gamma_\mu u + \bar{d} \gamma_\mu d}{2} + (e_u - e_d) \frac{\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d}{2} + e_s \bar{s} \gamma_\mu s \right), \end{aligned} \quad (11)$$

with  $e_u = 2/3e$ ,  $e_d = -1/3e$ , and  $e_s = -1/3e$  being the electric charges of  $u, d$ , and  $s$  quarks, respectively, we obtain

$$\begin{aligned} \frac{\mathcal{B}(f_1(1285) \rightarrow \phi\gamma)}{\mathcal{B}(f_1(1285) \rightarrow \rho^0\gamma)} &= \left( \frac{\langle \phi | e_s \bar{s} \gamma_\mu s | f_1(1285) \rangle}{\langle \rho | (e_u - e_d) (\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d) / 2 | f_1(1285) \rangle} \right)^2 \underbrace{\left( \frac{m_{f_1}^2 - m_\phi^2}{m_{f_1}^2 - m_\rho^2} \right)^3}_{\text{phase factor}} \\ &= \underbrace{\left( \frac{-e/3}{2e/3 + e/3} \right)^2}_{\text{EM factor}} \left( \frac{\langle \phi | \bar{s} \gamma_\mu s | f_1(1285) \rangle}{\langle \rho | (\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d) / 2 | f_1(1285) \rangle} \right)^2 \underbrace{\left( \frac{m_{f_1}^2 - m_\phi^2}{m_{f_1}^2 - m_\rho^2} \right)^3}_{\text{phase factor}} \end{aligned}$$

<sup>1</sup> Replacing the meson mass squared  $m^2$  by  $m$  throughout (4), we obtain  $\theta^{\text{lin}} = (23_{-23}^{+17})^\circ$ . The difference is negligible. Our result can be compared with that using  $\theta_{K_1} = -57^\circ$  into (4), one has  $\theta^{\text{quad}} = 52^\circ$ .

<sup>2</sup> In PDG [14], the mixing angle is defined as  $\alpha = \theta - \theta_i + \pi/2$ . Comparing it with our definition, we have  $\alpha = \pi/2 - \bar{\alpha}$ .

$$\approx \frac{4}{9} \left( \frac{m_\phi f_\phi}{m_\rho f_\rho} \right)^2 \tan^2 \bar{\alpha} \left( \frac{m_{f_1}^2 - m_\phi^2}{m_{f_1}^2 - m_\rho^2} \right)^3, \quad (12)$$

where  $f_1 \equiv f_1(1285)$ , and  $f_\phi$  and  $f_\rho$  are the decay constants of  $\phi$  and  $\rho$ , respectively. Here we have taken the single-pole approximation<sup>3</sup>:

$$\begin{aligned} \frac{\langle \phi | \bar{s} \gamma_\mu s | f_1(1285) \rangle}{\langle \rho | (\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d) / 2 | f_1(1285) \rangle} &\approx \frac{m_\phi f_\phi g_{f_1 \phi \phi}}{m_\rho f_\rho g_{f_1 \rho \rho} / \sqrt{2}} \frac{\sin \bar{\alpha}}{\cos \bar{\alpha} / \sqrt{2}} \\ &\approx \frac{m_\phi f_\phi}{m_\rho f_\rho} \times 2 \tan \bar{\alpha}. \end{aligned} \quad (13)$$

Using  $f_\rho = 209 \pm 1$  MeV,  $f_\phi = 221 \pm 3$  MeV [27], and the current data  $\mathcal{B}(f_1(1285) \rightarrow \phi \gamma) = (7.4 \pm 2.6) \times 10^{-4}$  and  $\mathcal{B}(f_1(1285) \rightarrow \rho^0 \gamma) = (5.5 \pm 1.3)\%$  [14] as inputs, we obtain  $\bar{\alpha} = \pm(15.8_{-4.6}^{+4.5})^\circ$ , i.e., two fold solution  $\theta = (19.4_{-4.6}^{+4.5})^\circ$  or  $(51.1_{-4.6}^{+4.5})^\circ$ . Combining with the analysis  $\theta = (0 \sim 40)^\circ$  given in Sec. II A, we thus find that  $\theta = (19.4_{-4.6}^{+4.5})^\circ$  is much preferred and can explain experimental observables well.

### III. MASS OF THE STRANGE QUARK

We proceed to evaluate the strange quark mass from the mass difference sum rules of the  $f_1(1285)$  and  $f_1(1420)$  mesons. We consider the following two-point correlation functions,

$$\Pi_{\mu\nu}(q^2) = i \int d^4 x e^{iqx} \langle 0 | T(j_\mu(x) j_\nu^\dagger(0)) | 0 \rangle = -\Pi_1(q^2) g_{\mu\nu} + \Pi_2(q^2) q_\mu q_\nu, \quad (14)$$

$$\Pi'_{\mu\nu}(q^2) = i \int d^4 x e^{iqx} \langle 0 | T(j'_\mu(x) j_\nu^\dagger(0)) | 0 \rangle = -\Pi'_1(q^2) g_{\mu\nu} + \Pi'_2(q^2) q_\mu q_\nu. \quad (15)$$

The interpolating currents satisfying the relations:

$$\langle 0 | j_\mu^{(\prime)}(0) | f_1^{(\prime)}(P, \lambda) \rangle = -i f_{f_1^{(\prime)}} m_{f_1^{(\prime)}} \epsilon_\mu^{(\lambda)}, \quad (16)$$

are

$$j_\mu = \cos \theta j_\mu^{(1)} + \sin \theta j_\mu^{(8)}, \quad (17)$$

$$j'_\mu = -\sin \theta j_\mu^{(1)} + \cos \theta j_\mu^{(8)}, \quad (18)$$

where

$$j_\mu^{(1)} = \frac{1}{\sqrt{3}} (\bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d + \bar{s} \gamma_\mu \gamma_5 s), \quad (19)$$

$$j_\mu^{(8)} = \frac{1}{\sqrt{6}} (\bar{u} \gamma_\mu \gamma_5 u + \bar{d} \gamma_\mu \gamma_5 d - 2 \bar{s} \gamma_\mu \gamma_5 s), \quad (20)$$

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<sup>3</sup> The following approximation was used in [26]:

$$\frac{\langle \phi | \bar{s} \gamma_\mu s | f_1(1285) \rangle}{\langle \rho | (\bar{u} \gamma_\mu u - \bar{d} \gamma_\mu d) / 2 | f_1(1285) \rangle} \approx 2 \tan \bar{\alpha}.$$

and we have used the short-hand notations for  $f_1 \equiv f_1(1285)$  and  $f_1' \equiv f_1(1420)$ . In the mass-less quark limit, we have  $\Pi_1 = q^2 \Pi_2$  and  $\Pi_1' = q^2 \Pi_2'$  if one neglects the axial-vector anomaly<sup>4</sup>. Here we focus on  $\Pi_1^{(\prime)}$  since it receives contributions only from axial-vector ( $^3P_1$ ) mesons, whereas  $\Pi_2^{(\prime)}$  contains effects from pseudoscalar mesons. The lowest-lying  $f_1^{(\prime)}$  meson contribution can be approximated via the dispersion relation as

$$\frac{m_{f_1^{(\prime)}}^2 f_{f_1^{(\prime)}}^2}{m_{f_1^{(\prime)}}^2 - q^2} = \frac{1}{\pi} \int_0^{s_0^{f_1^{(\prime)}}} ds \frac{\text{Im} \Pi_1^{(\prime)\text{OPE}}(s)}{s - q^2}, \quad (21)$$

where  $\Pi_1^{(\prime)\text{OPE}}$  is the QCD operator-product-expansion (OPE) result of  $\Pi_1^{(\prime)}$  at the quark-gluon level [20], and  $s_0^{f_1^{(\prime)}}$  is the threshold of the higher resonant states. Note that the subtraction terms on the right-hand side of (21), which are polynomials in  $q^2$ , are neglected since they have no contributions after performing the Borel transformation. The four-quark condensates are expressed as

$$\langle 0 | \bar{q} \Gamma_i \lambda^a q \bar{q} \Gamma_i \lambda^a q | 0 \rangle = -a_2 \frac{1}{16N_c^2} \text{Tr}(\Gamma_i \Gamma_i) \text{Tr}(\lambda^a \lambda^a) \langle \bar{q} q \rangle^2, \quad (22)$$

where  $a_2 = 1$  corresponds to the vacuum saturation approximation. In the present work, we have  $\Gamma = \gamma_\mu$  and  $\gamma_\mu \gamma_5$ , for which we allow the variation  $a_2 = -2.9 \sim 3.1$  [9, 28, 29]. For  $\Pi_1^{(\prime)\text{OPE}}$ , we take into account the terms with dimension  $\leq 6$ , where the term with dimension=0 ( $D=0$ ) is up to  $\mathcal{O}(\alpha_s^3)$ , with  $D=2$  (which is proportional to  $m_s^2$ ) up to  $\mathcal{O}(\alpha_s^2)$  and with  $D=4$  up to  $\mathcal{O}(\alpha_s^2)$ . Note that such radiative corrections for terms can read from [30–32]. We do not include the radiative correction to the  $D=6$  terms since all the uncertainties can be lumped into  $a_2$ . Note that such radiative corrections for terms with dimensions=0 and 4 are the same as the vector meson case and can read from [30, 31].

Further applying the Borel (inverse-Laplace) transformation,

$$\mathbf{B}[f(q^2)] = \lim_{\substack{n \rightarrow \infty \\ -q^2 \rightarrow \infty \\ -q^2/n^2 = M^2 \text{ fixed}}} \frac{1}{n!} (-q^2)^{n+1} \left[ \frac{d}{dq^2} \right]^n f(q^2), \quad (23)$$

to both sides of (21) to improve the convergence of the OPE series and further suppress the contributions from higher resonances, the sum rules thus read

$$\begin{aligned} f_{f_1}^2 m_{f_1}^2 e^{-m_{f_1}^2/M^2} &= \int_0^{s_0^{f_1}} \frac{s ds e^{-s/M^2}}{4\pi^2} \left[ 1 + \frac{\alpha_s(\sqrt{s})}{\pi} + F_3 \frac{\alpha_s^2(\sqrt{s})}{\pi^2} + (F_4 + F_4' \cos^2 \theta) \frac{\alpha_s^3(\sqrt{s})}{\pi^3} \right] \\ &\quad - (\cos \theta - \sqrt{2} \sin \theta)^2 [\bar{m}_s(\mu_o)]^2 \int_0^{s_0^{f_1}} ds \frac{1}{2\pi^2} e^{-s/M^2} \left[ 1 + \left( H_1 \ln \frac{s}{\mu_o^2} + H_2 \right) \frac{\alpha_s(\mu_o)}{\pi} \right] \end{aligned}$$

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<sup>4</sup> Considering the anomaly, the singlet axial-vector current is satisfied with

$$\partial^\mu j_\mu^{(1)} = \frac{1}{\sqrt{3}} (m_u \bar{u} u + m_d \bar{d} d + m_s \bar{s} s) + \frac{3\alpha_s}{4\pi} G \tilde{G}$$

$$\begin{aligned}
& + \left( H_{3a} \ln^2 \frac{s}{\mu_o^2} + H_{3b} \ln \frac{s}{\mu_o^2} + H_{3c} - \frac{H_{3a}\pi^2}{3} \right) \left( \frac{\alpha_s(\mu_o)}{\pi} \right)^2 \Big] \\
& - \frac{1}{12} \left( 1 - \frac{11}{18} \frac{\alpha_s(M)}{\pi} \right) \langle \frac{\alpha_s}{\pi} G^2 \rangle \\
& - \left[ \frac{4}{27} \frac{\alpha_s(M)}{\pi} + \left( -\frac{257}{486} + \frac{4}{3} \zeta(3) - \frac{2}{27} \beta_1 \gamma_E \right) \frac{\alpha_s^2(M)}{\pi^2} \right] \sum_{q_i \equiv u, d, s} \langle \bar{m}_i \bar{q}_i q_i \rangle \\
& + \frac{1}{3} (\sqrt{2} \cos \theta + \sin \theta)^2 \left[ 2a_1 \bar{m}_q \langle \bar{q}q \rangle - \frac{352\pi\alpha_s}{81M^2} a_2 \langle \bar{q}q \rangle^2 \right] \\
& + \frac{1}{3} (\cos \theta - \sqrt{2} \sin \theta)^2 \left[ 2a_1 \bar{m}_s \langle \bar{s}s \rangle - \frac{352\pi\alpha_s}{81M^2} a_2 \langle \bar{s}s \rangle^2 \right], \tag{24}
\end{aligned}$$

$$\begin{aligned}
f_{f_1'}^2 m_{f_1'}^2 e^{-m_{f_1'}^2/M^2} &= \int_0^{s_0'} \frac{s ds e^{-s/M^2}}{4\pi^2} \left[ 1 + \frac{\alpha_s(\sqrt{s})}{\pi} + F_3 \frac{\alpha_s^2(\sqrt{s})}{\pi^2} + (F_4 + F_4' \sin^2 \theta) \frac{\alpha_s^3(\sqrt{s})}{\pi^3} \right] \\
& + (\sin \theta + \sqrt{2} \cos \theta)^2 [\bar{m}_s(\mu_o)]^2 \int_0^{s_0'} ds \frac{1}{2\pi^2} e^{-s/M^2} \left[ 1 + \left( H_1 \ln \frac{s}{\mu_o^2} + H_2 \right) \frac{\alpha_s(\mu_o)}{\pi} \right. \\
& \quad \left. + \left( H_{3a} \ln^2 \frac{s}{\mu_o^2} + H_{3b} \ln \frac{s}{\mu_o^2} + H_{3c} - \frac{H_{3a}\pi^2}{3} \right) \left( \frac{\alpha_s(\mu_o)}{\pi} \right)^2 \right] \\
& - \frac{1}{12} \left( 1 - \frac{11}{18} \frac{\alpha_s(M)}{\pi} \right) \langle \frac{\alpha_s}{\pi} G^2 \rangle \\
& - \left[ \frac{4}{27} \frac{\alpha_s(M)}{\pi} + \left( -\frac{257}{486} + \frac{4}{3} \zeta(3) - \frac{2}{27} \beta_1 \gamma_E \right) \frac{\alpha_s^2(M)}{\pi^2} \right] \sum_{q_i \equiv u, d, s} \langle \bar{m}_i \bar{q}_i q_i \rangle \\
& + \frac{1}{3} (\sqrt{2} \sin \theta - \cos \theta)^2 \left[ 2a_1 \bar{m}_q \langle \bar{q}q \rangle - \frac{352\pi\alpha_s}{81M^2} a_2 \langle \bar{q}q \rangle^2 \right] \\
& + \frac{1}{3} (\sin \theta + \sqrt{2} \cos \theta)^2 \left[ 2a_1 \bar{m}_s \langle \bar{s}s \rangle - \frac{352\pi\alpha_s}{81M^2} a_2 \langle \bar{s}s \rangle^2 \right], \tag{25}
\end{aligned}$$

where

$$F_3 = 1.9857 - 0.1153n_f \simeq 1.6398 \quad \text{for } n_f = 3,$$

$$F_4 = -6.6368 - 1.2001n_f - 0.0052n_f^2 \simeq -10.2839 \quad \text{for } n_f = 3,$$

$$F_4' = -1.2395\Delta,$$

$$H_1 = -\frac{8}{81}\beta_1^2 = -2, \quad H_2 = \frac{2}{9}\beta_2 + 4\beta_2 \left( \frac{\gamma_1}{\beta_1} - \frac{\gamma_2}{\beta_2} \right) - \frac{8}{9}\beta_1^2 - 4\beta_1 \simeq 3.6667,$$

$$H_{3a} = 4.2499, \quad H_{3b} = -23.1667, \quad H_{3c} = 29.7624,$$

$$\begin{aligned}\overline{m}_q \langle \bar{q}q \rangle &\equiv \frac{1}{2} (\overline{m}_u \langle \bar{u}u \rangle + \overline{m}_d \langle \bar{d}d \rangle), \quad \langle \bar{q}q \rangle^2 \equiv \frac{1}{2} (\langle \bar{u}u \rangle^2 + \langle \bar{d}d \rangle^2), \\ a_1 &= 1 + \frac{7}{3} \frac{\alpha_s(M)}{\pi} + \left( \frac{85}{6} - \frac{7}{6} \beta_1 \gamma_E \right) \frac{\alpha_s^2(M)}{\pi^2},\end{aligned}\tag{26}$$

with  $\beta_1 = (2n_f - 33)/6$ ,  $\beta_2 = (19n_f - 153)/12$ ,  $\gamma_1 = 2$ ,  $\gamma_2 = 101/12 - 5n_f/18$ , and  $n_f = 3$  being the number of flavors and  $\Delta = 1$ , and 0 for  $f_1$  (singlet) and  $f_8$  (octet), respectively [32]. In the calculation the coupling constant  $\alpha_s(\sqrt{s})$  in Eqs. (24) and (25) can be expanded in powers of  $\alpha_s(M)$ :

$$\begin{aligned}\frac{\alpha_s(\sqrt{s})}{\pi} &= \frac{\alpha_s(M)}{\pi} + \frac{1}{2} \beta_1 \ln \frac{s}{M^2} \left( \frac{\alpha_s(M)}{\pi} \right)^2 + \left( \frac{1}{2} \beta_2 \ln \frac{s}{M^2} + \frac{1}{4} \beta_1^2 \ln^2 \frac{s}{M^2} \right) \left( \frac{\alpha_s(M)}{\pi} \right)^3 \\ &+ \left( \frac{\beta_3}{2} \ln \frac{s}{M^2} + \frac{5}{8} \beta_1 \beta_2 \ln^2 \frac{s}{M^2} + \frac{1}{8} \beta_1^3 \ln^3 \frac{s}{M^2} \right) \left( \frac{\alpha_s(M)}{\pi} \right)^4 + \dots,\end{aligned}\tag{27}$$

where  $\beta_3 \simeq -20.1198$ . Using the renormalization-group result for the  $m_s^2$  term given in [31], we have expanded the contribution to the order  $\mathcal{O}(\alpha_s^2 m_s^2)$  at the subtraction scale  $\mu_o^2 = 2 \text{ GeV}^2$  for which the series has better convergence than at the scale  $1 \text{ GeV}^2$ ; however, the convergence of the series has no obvious change if using a higher reference scale. As in the case of flavor-breaking  $\tau$  decay, the  $D = 2$  series converges slowly; nevertheless, we have checked that this term, which intends to make the output  $m_s$  to be smaller in the fit, is suppressed due to the fact that the mass sum rules for  $f_1(1285)$  and  $f_1(1420)$  are obtained by applying the differential operator  $M^4 \partial \ln / \partial M^2$  to both sides of (24) and (25), respectively. Nevertheless, the differential operator will instead make the  $D=4$  term containing  $m_s \langle \bar{s}s \rangle$  become much more important than the  $m_s^2$  term in determining the  $f_1(1285)$ - $f_1(1420)$  mass difference although they are the same order in magnitude.

In the numerical analysis, we shall use  $\Lambda_{\text{QCD}}^{(3)\text{NLO}} = 0.360 \text{ GeV}$ , corresponding to  $\alpha_s(1\text{GeV}) = 0.495$ ,  $\Lambda_{\text{QCD}}^{(4)\text{NLO}} = 0.313 \text{ GeV}$ , and the following values (at the scale  $\mu = 1 \text{ GeV}$ ) [9, 28, 29, 33]:

$$\begin{aligned}\langle \frac{\alpha_s}{\pi} G_{\mu\nu}^a G^{a\mu\nu} \rangle &= (0.009 \pm 0.007) \text{ GeV}^4, \\ \langle \overline{m}_q \bar{q}q \rangle &= -f_{\pi^+}^2 m_{\pi^+}^2 / 4, \\ \langle \bar{q}q \rangle^2 &\simeq (-0.247)^6 \text{ GeV}^6, \\ \langle \bar{s}s \rangle &= (0.30 \sim 1.3) \langle \bar{q}q \rangle, \\ a_2 &= -2.9 \sim 3.1,\end{aligned}\tag{28}$$

where the value of  $\langle \bar{q}q \rangle^2$  corresponds to  $(m_u + m_d)(1\text{GeV}) \simeq 11 \text{ MeV}$ , and we have cast the uncertainty of  $\langle \bar{q}q \rangle^2$  to  $a_2$  in the  $D = 6$  term. We do not consider the isospin breaking effect between  $\langle \bar{u}u \rangle$  and  $\langle \bar{d}d \rangle$  since  $\langle \bar{d}d \rangle / \langle \bar{u}u \rangle - 1 \approx -0.007$  [34] is negligible in the present analysis. The threshold is allowed by  $s_0^{f_1} = 2.70 \pm 0.15 \text{ GeV}^2$  and determined by the maximum stability of the mass sum rule. For an estimate on the threshold difference, we parametrize in the form  $(\sqrt{s_0^{f_1'}} - \sqrt{s_0^{f_1}}) / \sqrt{s_0^{f_1}} = \delta \times (m_{f_1'} - m_{f_1}) / m_{f_1}$ , with  $\delta = 1.0 \pm 0.3$ . In other words, we assign a 30% uncertainty to the default value. We search for the allowed solutions for strange quark mass and the singlet-octet mixing angle  $\theta$  under the following constraints: (i) Comparing with the observables, the errors for the mass sum rule results of the  $f_1(1285)$  and  $f_1(1420)$  in the Borel



TABLE I. The fitting results in the  $f_1(1284)$ - $f_1(1420)$  mass difference sum rules. In fit II, we have taken the average of the recent lattice results for  $m_s$ , which is rescaled to 1 GeV as the input.

	$m_s(1 \text{ GeV})$	$\langle \bar{s}s \rangle / \langle \bar{u}u \rangle$	$\langle (\alpha_s/\pi) G^2 \rangle$	$a_2$
Fit I	$106.3 \pm 35.1$	$0.56 \pm 0.25$	$0.0106 \pm 0.0042$	$0.89 \pm 0.62$
Fit II	$[124.7 \pm 1.3]$	$0.41 \pm 0.09$	$0.0108 \pm 0.0037$	$0.95 \pm 0.45$

window  $0.9 \text{ GeV}^2 \leq M^2 \leq 1.3 \text{ GeV}^2$  are constrained to be less than 3% on average. In this Borel window, the contribution originating from higher resonances (and the continuum), modeled by

$$\frac{1}{\pi} \int_{s_0^{f_1}}^{\infty} ds e^{-s/M^2} \text{Im}\Pi_1^{(\prime)\text{OPE}}(s), \quad (29)$$

is about less than 40% and the highest OPE term (with dimension six) at the quark level is no more than 10%. (ii) The deviation between the  $f_1(1420)$ - $f_1(1285)$  mass difference sum rule result and the central value of the data [14] is within  $1\sigma$  error:  $|(m_{f_1'} - m_{f_1})_{\text{sum rule}} - 144.6 \text{ MeV}| \leq 1.5 \text{ MeV}$ . The detailed results are shown in Table 1. We also check that if by further enlarging the uncertainties of  $s_0^{f_1}$  and  $\delta$ , *e.g.* 25%, the changes of results can be negligible. We obtain the strange quark mass with large uncertainty:  $m_s(1 \text{ GeV}) = 106.3 \pm 35.1 \text{ MeV}$  (i.e.  $m_s(2 \text{ GeV}) = 89.5 \pm 29.5 \text{ MeV}$ ) and  $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle = 0.56 \pm 0.25$  corresponding to  $\theta = (19.4_{-4.6}^{+4.5})^\circ$ , where the values and  $m_s$  and  $\langle \bar{s}s \rangle$  are strongly correlated.

Further accounting for the average of the recent lattice results [15–17]:  $m_s(2 \text{ GeV}) = 93.6 \pm 1.0 \text{ MeV}$  and using the  $\theta$  value that we have obtained as the inputs, we get  $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle = 0.41 \pm 0.09$  which is less than one and in contrast to the Schwinger-Dyson equation approach in [6] where the ratio was obtained as  $(1.0 \pm 0.2)^3$ . Our prediction is consistent with the QCD sum rule result of studying the scalar/pseudoscalar two-point function in [35] where the authors obtained  $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle = 0.4 \sim 0.7$ , depending on the value of the strange quark mass.

#### IV. SUMMARY

We have adopted two different strategies for determining the mixing angle  $\theta$ : (i) Using the Gell-Mann-Okubo mass formula and the  $K_1(1270)$ - $K_1(1400)$  mixing angle  $\theta_{K_1} = (-34 \pm 13)^\circ$  which was extracted from the data for  $\mathcal{B}(B \rightarrow K_1(1270)\gamma)$ ,  $\mathcal{B}(B \rightarrow K_1(1400)\gamma)$ ,  $\mathcal{B}(\tau \rightarrow K_1(1270)\nu_\tau)$ , and  $\mathcal{B}(\tau \rightarrow K_1(1420)\nu_\tau)$ , the result is  $\theta = (23_{-23}^{+17})^\circ$ . (ii) On the other hand, from the analysis of the ratio of  $\mathcal{B}(f_1(1285) \rightarrow \phi\gamma)$  and  $\mathcal{B}(f_1(1285) \rightarrow \rho^0\gamma)$ , we have  $\bar{\alpha} = \theta_i - \theta = \pm(15.8_{-4.6}^{+4.5})^\circ$ , i.e.,  $\theta = (19.4_{-4.6}^{+4.5})^\circ$  or  $(51.1_{-4.6}^{+4.5})^\circ$ . Combining these two analyses, we deduce the mixing angle  $\theta = (19.4_{-4.6}^{+4.5})^\circ$ .

We have estimated the strange quark mass and strange quark condensate from the analysis of the  $f_1(1420)$ - $f_1(1285)$  mass difference QCD sum rule. We have expanded the OPE series up to dimension six, where the term with dimension zero is up to  $\mathcal{O}(\alpha_s^3)$ , with dimension=2 up to  $\mathcal{O}(m_s^2\alpha_s^2)$  and with dimension=4 terms up to  $\mathcal{O}(\alpha_s^2)$ . Further using the average of the recent lattice results and the  $\theta$  value that we have obtained as the inputs, we get  $\langle \bar{s}s \rangle / \langle \bar{u}u \rangle = 0.41 \pm 0.09$ .

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